Two Counterexamples in the Logic of Dynamic Topological Systems.

Sergey Slavnov
Cornell University,
Department of Mathematics, Malott Hall, Ithaca, NY 14853-4201, USA

serge@math.cornell.edu, slavnovs@slavnov.ips.ras.ru

Abstract

The classical Tarski theorem on topological semantics of modal logic states that the logic **S4** is complete in $\mathbf{R}^n$ for each n. Recently several authors have considered logics of dynamic topological systems, which is a topological space and a function on it. In [1] a bimodal logic **S4C** was introduced and proven to be complete with respect to the class of all continuous dynamic systems. A number of polymodal logics for dynamic topological systems were considered in [3, 4, 5]. In [5] a modal logic of dynamic systems with homeomorphisms was axiomatized and proven to enjoy the analogue of the Tarski theorem. In this note it is shown that the analogue of the Tarski theorem does not hold for **S4C**, a question posed by Artemov and Nerode. In the language with an iteration of the dynamic system function, we also construct an **R**-valid formula that does not hold in the logic of dynamic systems with homeomorphisms. This proves that the analogue of the Tarski theorem does not hold for the logic of homeomorphisms with iterations.

1 Introduction

The topological semantics for the propositional modal language interprets $\Box$ as the interior operation. In this semantics, each propositional symbol is evaluated by a subset of X and boolean connectives behave in the usual set theoretic way. A formula is valid in a given X if its evaluation coincides with the whole of X (cf., [6]).

A *dynamic system* is a pair (X, f), where X is a topological space and f is a total function from X to X. A function's behavior can be modeled by a temporal modality "next", which we will denote as $[a]$. The modality $[a]$ is interpreted in a dynamic system (X, f) as the inverse image of a set under f.

The logic of continuous dynamic systems **S4C** in the language $L_{\Box,a}$ with modalities $\Box$ for interior and $[a]$ for "next" was presented and proven complete in [1] (a fragment of **S4C** was axiomatized independently in [3, 5]). The completeness proof from [1], however did not provide a countermodel in the reals $\mathbf{R}$ under the usual topology. Since the principal examples of dynamic systems came from $\mathbf{R}^n$, the question remained whether the Tarski theorem can be extended to **S4C**. In particular, Artemov and Nerode asked in private communications at Cornell University whether **S4C** is complete with respect to the class of continuous dynamic systems on $\mathbf{R}$. In this note we give a negative answer by providing a counterexample, discovered two years ago. While working on this note, the author also received a private communication from Johan van Benthem with another elegant counterexample.

Let H be the class of dynamic systems (X, f) whose function f is a homeomorphism. In [5], the logic $\mathbf{DTL_H}$ of H in a richer language $L_{\Box,a,*}$ with the third modality $*$ standing for the iteration of f was considered. This third modality is to be read as "from now on" and is interpreted by the intersection of all iterated inverse images. The next-interior fragment of $\mathbf{DTL_H}$ has been axiomatized and proven to be complete. It turned out that this fragment was also complete with respect to the topology on $\mathbf{R}$. We prove that the full logic of homeomorphisms with the iteration cannot be complete with respect to the topology on $\mathbf{R}$ by providing an example of a formula in the language $L_{\Box,a,*}$ that is valid on $\mathbf{R}^n$ and false in H. This shows that the Tarski theorem does not extend to the logic of H in the language $L_{\Box,a,*}$.

Acknowledgements I am grateful to Grigori Mints for his important remarks which helped to improve the text. I also would like to thank Johan van Benthem for his communication and his kind permission to include his counterexample in these notes. I am also grateful to Bryan Renne and Sergei Artemov who proofread this note and once more to Sergei Artemov for many useful suggestions.

2 Topological semantics and Tarski theorem

Definition 1 *Language $L_\Box$ consists of a countable set of propositional symbols, a constant $\bot$, binary connective $\to$, and unary operator $\Box$.*

Other Boolean connectives are definable in the usual way.

Let X be a topological space.

Definition 2 *A topological model (for language $L_\Box$) $\langle X, \| \ \| \rangle$ in X consists of a valuation $\| \ \|$ which assigns to any formula p a subset $\|p\| \subseteq X$ and satisfies the following conditions:*

$$\|\bot\| = \emptyset$$

$$\|A \to B\| = -\|A\| \cup \|B\|$$

$$\|\Box A\| = Int\|A\|$$

We say that a formula $\phi \in L_\Box$ is *satisfied* in a model if $\|\phi\| = X$. We say that ϕ is *valid* in X if it is satisfied in any topological model in X. At last we call a formula *topologically valid* if it is valid in any topological space.

The axiomatic system corresponding to this semantics is propositional **S4**.

Definition 3 *Logic* **S4** *contains following schemes:*

CP : *axioms of classical propositional logic in $L_\Box$*

$\Box T$: $\Box\phi \to \phi$

$\Box K$: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

$\Box 4$: $\Box\phi \to \Box\Box\phi$

and inference rules:

modus ponens $\qquad \dfrac{\phi, \phi \to \psi}{\psi}$

$\Box -$ *necessitation* : $\qquad \dfrac{\phi}{\Box\phi}$

Connections between this system and topological semantics are summarized in the following well-known completeness theorem (see [2]):

Theorem 1 *For any formula $\phi \in L_\Box$ the following are equivalent:*

(i) **S4** $\vdash \phi$;

(ii) ϕ is topologically valid;

(iii) ϕ is true in any finite topological space.

One may argue that finite topological spaces are rather exotic and wonder if this completeness result may be extended to some habitual spaces such as $\mathbf{R}^n$. Indeed, the Tarski theorem gives a positive answer to this question.

Theorem 2 (Tarski theorem [2]) *Let X be a metric space that is dense in itself. If a formula $\phi \in L_\Box$ is not derivable in* **S4** *then there exists a topological model in X refuting ϕ.*

Note that the real space is dense in itself and it follows that **S4** is complete over $\mathbf{R}^n$.

3 Topological Dynamic Logic S4C

In this section we recall the logic **S4C**, an extension of **S4** introduced and proven to be complete by Artemov, Davoren, and Nerode in [1]. Next we will show that the analogue of the Tarski theorem does not hold for **S4C**. We follow the notation of [1].

The language $L_{\Box,a}$ is the language of **S4** enriched with the modal operator $[a]$. Let X be a topological space.

Definition 4 *A (continuous) topological model (for $L_{\Box,a}$) on X consists of a (continuous) total function $f : X \to X$ and a valuation $\| \ \|$ which assigns to any formula $p \in L_{\Box,a}$ a subset $\|p\| \subseteq X$ and satisfies all the conditions of Definition 2 in addition to the following condition:*

$$\|[a]A\| = f^{-1}(\|A\|).$$

All terms *valid*, *satisfied*, etc. are defined in the same fashion as for **S4**.

From now on by topological model we shall mean a continuous topological model for $L_{\Box,a}$.

Definition 5 *Logic* **S4C** *contains axioms and rules of* **S4** *in $L_{\Box,a}$ and the following schemes:*

$[a]K: \ [a](\phi \to \psi) \to ([a]\phi \to [a]\psi)$

$[a]\neg: \ [a]\neg\phi \leftrightarrow \neg[a]\phi$

Cont: $[a]\Box\phi \leftrightarrow \Box[a]\Box\phi$

and the inference rule

$$[a] - necessitation: \qquad \frac{\phi}{[a]\phi}$$

The axiom **Cont** expresses continuity; it says precisely that the inverse image of an open set is open.

One can easily show the following:

Note 1 *In* **S4C** *connective* $[a]$ *commutes with all Boolean connectives.*

As in **S4**, we have the following theorem:

Theorem 3 *[1] For any formula* $\phi \in L_{\Box,a}$ *the following are equivalent:*

(i) $\mathbf{S4C} \vdash \phi$*;*

(ii) ϕ *is topologically valid;*

(iii) ϕ *is true in any finite topological space.*

4 Tarski theorem fails for S4C and R: a counterexample

We give a simple example of a formula not derivable in **S4C** but valid in **R**. Recall that $\diamond\phi$ is a short for $\neg\Box\neg\phi$. As is easy to see in the topological interpretation, $\diamond$ means 'the closure'.

Let the formula ϕ be defined by

$$\phi := \diamond\Box p \& \diamond\neg\Box p. \tag{1}$$

Note 2 *In any topological model the formula* ϕ *defined above denotes the boundary of an open set.*

Now let $\langle \mathbf{R}, f, \| \ \| \rangle$ be a topological model. Consider the formula

$$\psi := \Box[a]\phi \& [a]q \& \diamond [a]\neg q \tag{2}$$

where ϕ was defined in (1).

Lemma 1 $\|\psi\| = \emptyset$.

Proof Suppose $x \in \|\psi\|$. The first term of conjunction (2) says that for some sufficiently small open interval U containing x and for any $y \in U$, the image $f(y) \in \|\phi\|$ and thus $f(U) \subseteq \|\phi\|$. So by Note 2, the image $f(U)$ is the boundary of an open set. Now U is connected and the continuous image of a connected set is connected. Since any boundary of an open set in $\mathbf{R}$ is a discrete collection of points, the only connected boundaries in $\mathbf{R}$ are singletons. Thus U is mapped by f to a single point, i.e. $f(U) = \{f(x)\}$.

The second term of conjunction (2) says that $f(x) \in \|q\|$ and hence $f(U) \subseteq \|q\|$. But the third term says that in any neighborhood of x and, in particular, in U, there exists a point y such that $f(y) \notin \|q\|$. This means, in particular, that $f(x) \neq f(y)$ and $f(U) \neq \{f(x)\}$. So we have a contradiction.

Thus $\neg\psi$ is valid in $\mathbf{R}$. It is not hard however to construct a topological model where $\|\psi\|$ is nonempty.

For example, take $\langle \mathbf{R}^2, f, \| \ \| \rangle$ where

$$f(x, y) = (x, 0),$$

$$\|p\| = \{(x, |y|) |\ x, y \in \mathbf{R}\},$$

and

$$\|q\| = \{(0, 0)\}.$$

Then

$$\|\psi\| = \{(0, y) |\ y \in \mathbf{R}\}$$

In view of topological completeness of **S4C** this shows that $\neg\psi$ is not derivable in **S4C**.

Thus one cannot generalize the Tarski theorem to **S4C** and the real line. Moreover, it is clear that the same argument works for any one-dimensional topological manifold.

A similar counterexample has recently been found independently. While working on these notes, the author received a private communication with another counterexample from Johan van Benthem together with his kind permission to reproduce it. That counterexample was obtained by van Benthem in collaboration with Philip Kremer.

Van Benthem considers the following formula:

$$\tau := ([a]\Box \diamond \alpha \& \diamond [a]\beta) \to ([a]\beta \cup \diamond [a]\alpha). \tag{3}$$

We can write

$$\neg\tau = [a]\Box \diamond \alpha \& \diamond [a]\beta \& \neg [a]\beta \& \Box \neg [a]\alpha \tag{4}$$

which is **S4C**-equivalent to

$$\Box [a](\Box \diamond \alpha \& \neg\alpha) \& \diamond [a]\beta \& [a]\neg\beta = \Box [a]\phi' \& \diamond [a]\beta \& [a]\neg\beta. \tag{5}$$

(we applied the **Cont** principle to the first term in conjunction (4) and used Note 1 which says that $[a]$ commutes with all Boolean connectives.) Comparing (5) with (2) we see that after substituting $\neg\beta$ for q, the formulas ψ and $\neg\tau$ differ by only the subformulas ϕ and ϕ'. It is not hard to see that for any interpretation in $\mathbf{R}$, the formula ϕ' denotes a disconnected set just as the formula ϕ defined in (1). Thus the geometry of this counterexample is essentially the same as the one considered in Lemma 1 above. One should note however that unlike ϕ given in (1), the formula ϕ' denotes a more general disconnected subset of $\mathbf{R}$ than the boundary of an open subset. For example, if α denotes the set of rational numbers then ϕ' denotes the set of irrational numbers.

5 Logic of homeomorphisms

In view of the counterexamples above, it may be reasonable to narrow the class of functions so as to obtain a dynamic version of the Tarski theorem. The next natural class is the class of homeomorphisms. In fact, this class deserves attention in its own right since the evolution of a dynamical system is usually given by homeomorphisms.

Topological models with homeomorphisms, and more generally open maps, satisfy the following principle:

Open $$\Box[a]\phi \to [a]\Box\phi,$$

see [1], [5].

Kremer, Mints, and Rybakov show in [5] that the logic of homeomorphisms is obtained from **S4C** precisely by adding the **Open** principle. They also show that the Tarski theorem is enjoyed by this new system. They further consider a richer language $L_{\Box,a,*}$ containing the modal operator $*$, "from now on".

Definition 6 *A topological model (for $L_{\Box,a,*}$) in a topological space X consists of a total function $f : X \to X$ and a valuation $\| \ \|$ which assigns to any formula $p \in L_{\Box,a,*}$ a subset $\|p\| \subseteq X$ and satisfies all the conditions of Definition* 4 *and the following condition:*

$$\|{*A}\| = \bigcap_{n\geq 0} f^{-n}(\|A\|).$$

From now on by a topological model we mean a continuous topological model $\langle X, f, \| \ \|\rangle$ for $L_{\Box,a,*}$ such that f is a homeomorphism.

Definition 7 *The logic* $\mathbf{DTL_H}$ *consists of all formulas in $L_{\Box,a,*}$ which are valid in all topological models.*

In the last section we show that the Tarski theorem does not extend from the next-interior fragment to full $\mathbf{DTL_H}$.

6 The Tarski theorem fails for DTL_H

The operator $*$ is essentially a universal quantifier. Therefore, the Tarski theorem fails for $\mathbf{DTL_H}$ in the same way as it fails for first-order **S4** (see [6], XI.11). In fact, the counterexample is the same.

Recall that for any topological interpretation the formula

$$\epsilon := p\&\neg\Box\Diamond p \tag{6}$$

denotes a nowhere dense set, i.e. a set whose closure has empty interior, see [6], III.11.

Let $\langle \mathbf{R}^n, f, \| \ \|\rangle$ be a topological model with f a homeomorphism.

Let ψ be given by

$$\psi := \Box\neg * \neg\epsilon. \tag{7}$$

Lemma 2 *The formula ψ denotes the empty set.*

Proof We have $\|\neg * \neg\epsilon\| = \bigcup_{n\geq 0} f^{-n}(\|\epsilon\|)$. Since f is a homeomorphism and $\|\epsilon\|$ is nowhere dense, it follows that $f^{-n}(\|\epsilon\|)$ is nowhere dense for every n. In $\mathbf{R^n}$, the union of a countable collection of nowhere dense sets is itself nowhere dense; in particular, the interior is empty.

Note that this argument works for any complete metric space.

However there exists a topological model where ψ is satisfied.

Consider the space $\mathbf{Z}$ of integers equipped with the order topology, and let W be the disjoint union of a countable collection of copies of $\mathbf{Z}$:

$$W = \bigcup_{m\in\mathbf{Z}} \mathbf{Z}.$$

That is, W is the set of all pairs of integers (m,n) whose topology is generated by basic neighborhoods $O(m,n) = \{(m,n')|\ n \leq n'\}$.

Let $f : W \to W$ be given by

$$f(m,n) = (m+1, n).$$

Obviously f is a homeomorphism. Finally, let

$$\|p\| = \{(m,n)|\ m > n\}.$$

Note that $\|p\|$ is closed in the topology of W and it contains no basic open neighborhood; hence the interior of $\|p\|$ is empty. Thus $\|\neg\Box \diamond p\| = W$ and $\|\epsilon\| = \|p\|$. Note also that for any $(m,n) \in W$ there is an N sufficiently large such that the image $f^N(m,n)$ lies in $\|p\|$. This yields $\|\neg * \neg\epsilon\| = W$, and the statement follows.

References

[1] S.N. Artemov, J.M. Davoren and A.Nerode, "Modal Logics and Topological Semantics for Hybrid Systems", Technical report MSI 97-05, Cornell University, 1997.

[2] J.McKinsey, A.Tarski, The Algebra of Topology, Ann.Math, 45, 1944, 141-191

[3] Philip Kremer, "Temporal logic over S4: An axiomatizable fragment of dynamic topological logic.", Bulletin of Symbolic logic 3(1997) 375-376

[4] Philip Kremer and Grigori Mints, “Dynamic topological logic.”, Bulletin of Symbolic logic 3(1997) 371-372

[5] Philip Kremer, Grigori Mints and Vladimir Rubakov, “Axiomatizing the next-interior fragment of dynamic topological logic.”, Bulletin of Symbolic logic 3(1997) 376-376

[6] Helena Rasiowa & Roman Sikorski, “The Mathematics of metamathematics” Panstowowe Wyfavnictwo Naukowe, Warsaw, 1963.